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Electromagnetic Two-Body Problem*

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The circular motion of two classical point charges interacting through their time-symmetric electromagnetic fields is determined with complete relativistic rigor. Expressions are obtained for the total energy and angular momentum of this two-body system, which include the contributions of the field. This work is a preliminary to a calculation, undertaken jointly with Schlosser and now in progress, of the relativistic electromagnetic interaction of two classical particles with charge, spin, and magnetic moment, and of the quantization of this classical two-body system. The purpose of the investigation is to check the possibility that some elementary particles may be atoms composed of other elementary particles in states where the principal interaction is due, not to the Coulomb force between point charges, but to forces where the intrinsic magnetic moments of the particles play an essential role and must not be considered as small perturbations. A very rough, but very simple, estimate by Corben indicates that this possibility is worth exploring.

1. INTRODUCTION

THE following simple observation has been made by Corben[1]:

Consider a magnetic dipole μ at rest and an electric charge e, of mass m, circling in the equatorial plane of μ under the Lorentz force $e\mathbf{v}\times\mathbf{H}$ (Fig. 1). The relativistic equation of motion of the point charge is

$$\frac{mv^2}{r(1-v^2)^{1/2}}=\frac{ev\mu}{r^3}. \tag{1.1}$$

Here r is the radius of the circular orbit and v the speed of e. Here, and throughout, units have been chosen so that the speed of light is unity ($c=1$). The mechanical angular momentum of the moving point charge is

$$L=mvr/(1-v^2)^{1/2}, \tag{1.2}$$

and its mechanical energy is

$$E=m(1-v^2)^{-1/2}. \tag{1.3}$$

Assume that e, m, μ are the electric charge, mass, and magnetic dipole moment of the electron, so that $\mu=e\hbar/2m$. Perform naive Bohr quantization by equating the mechanical angular momentum of e to an integer multiple of $\hbar$:

$$mvr/(1-v^2)^{1/2}=n\hbar. \tag{1.4}$$

Then the quantized motion, determined by Eqs. (1.1) and (1.4), is given by

$$r=\frac{1}{n}\frac{e^2}{2m}=\frac{1}{n}\times1.4\times10^{-13}\text{ cm}, \tag{1.5}$$

$$v/(1-v^2)^{1/2}=2n^2\hbar/e^2=274n^2, \tag{1.6}$$

$$E\cong2n^2m\hbar/e^2=274n^2m. \tag{1.7}$$

Thus, the resulting motion is highly relativistic and the orbits are of the order of nuclear dimensions. This suggests the possibility that some elementary particles may be atoms composed of other elementary particles in states where the principal interaction is due, not to the Coulomb force between point charges, but to electromagnetic forces where the intrinsic magnetic moments of the particles play an essential role and must not be considered as small perturbations.

In order to see if there is any reality in this possibility, one would like to study the quantum mechanical two-body problem of particles, each with charge, spin, and magnetic moment, moving in one another's electromagnetic field. One would then search for highly relativistic stationary states, if they exist. Unfortunately, we do not know how to attack a highly relativistic two-body problem within the framework of quantum electrodynamics. We therefore propose to do a simpler calculation, a relativistically rigorous classical calculation, to be followed by Bohr-type quantization.

The procedure is outlined in Table I, and will now be discussed briefly: (1) We wish to find stationary states, or periodic motions, for a system of two particles in electromagnetic interaction, with all relativistic effects (such as retardation) rigorously included. (2) We wish to find the total energy and total angular momentum of the stationary states of the system which consists of two particles and the electromagnetic field. The contributions of the field may be important in the highly relativistic states for which we are searching, and are to be included in the expressions for energy and angular momentum. (3) We apply Bohr quantization to the system.

(1) In order to have any stationary states at all, it seems necessary to assume that the electromagnetic interaction between the particles is time-symmetric, i.e., half-retarded plus half-advanced, rather than purely retarded. This is true for the case of two point charges

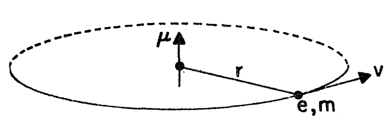

FIG. 1. Point charge in magnetic dipole field.

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[1] H. C. Corben (unpublished).

in circular orbits, and is probably true in other cases. We, therefore, assume that each particle moves in the time-symmetric electromagnetic field of the other. (2a) This choice of the time-symmetric interaction results in another benefit. It is possible to derive the equations of motion from a single action principle for a system of interacting particles, where the electromagnetic field variables have been eliminated from the action. This is called a Fokker or Wheeler-Feynman action principle.[2] (2b) Once a Fokker action principle is known, there are standard methods for deriving, from the Lorentz invariance of the action, conservation laws for linear momentum and energy and for angular momentum.[3] The conservation laws yield finite expressions for the energy and angular momentum of the system, which automatically include contributions of the electromagnetic field, and which can be used to study the quantized states of the system. From the point of view of classical field theory, a Fokker action principle provides a systematic relativistic algorithm for the subtraction of infinite contributions by the singularities of the field to energy, momentum, and angular momentum.

Schlosser and the author have found a Fokker action principle for the electromagnetic interaction of classical particles with charge, spin, and magnetic moment. The rest of the program outlined in Table I is then a matter of reasonably straight forward calculations. These calculations are now in progress.

In this paper a preliminary calculation is reported. The program of Table I is carried out fully for a simple model case, the problem of two point charges in circular motion.

TABLE I. The relativistic two-body problem.

Rigorous relativistic two-body problem (Classical)		Method
1. Find *stationary states* (periodic motion)	← 1.	*Time-symmetric interaction* ($\frac{1}{2}$ retarded$+\frac{1}{2}$ advanced) ↑
	2(a).	*Fokker* (*or Wheeler-Feynman*) *type action principle* ↓ (Lorentz invariance of action)
2. Compute *total energy and total angular momentum*, including contributions of field.	← 2(b).	*Conservation Laws*. Finite expressions for energy, momentum and angular momentum of system.
3. Apply *naive Bohr* (correspondence principle) *quantization*.		

2. THE ACTION PRINCIPLE

The Lorentz equations of motion, without radiation reaction force, of a charge e moving in an external electromagnetic field $F_{\mu\nu}$ are

$$m\ddot{x}^{\mu}=e\dot{x}^{\nu}F_{\nu}{}^{\mu}. \tag{2.1}$$

Here Greek suffixes range and sum over 1, 2, 3, 4, m is the mass of the moving charge, $x^{\mu}(s)$ its world line. A dot denotes differentiation with respect to the proper time s, given by

$$ds^2=\eta_{\mu\nu}dx^{\mu}dx^{\nu}=-(dx^1)^2-(dx^2)^2-(dx^3)^2+(dx^4)^2, \tag{2.2}$$

and the Minkowski metric tensor $\eta_{\mu\nu}$ is used to raise and lower tensor suffixes.

The equations of motion (2.1) can be obtained from the well-known variational principle

$$\delta\left[m\int ds+e\int A_{\mu}\dot{x}^{\mu}ds\right]=0, \tag{2.3}$$

where the world line of the charge is varied, with end points held fixed, and where A_{μ} are the electromagnetic potentials so that

$$F_{\mu\nu}=(\partial A_{\mu}/\partial x^{\nu})-(\partial A_{\nu}/\partial x^{\mu}). \tag{2.4}$$

Let the external electromagnetic field acting on e be the time-symmetric field produced by another point charge $\bar{e}$ with world line $\bar{x}^{\mu}(\bar{s})$. It is given by the half-retarded plus half-advanced Lienard-Wiechert potential

$$A_{\mu}(x)=\bar{e}\int\delta((x-\bar{x})^2)\dot{\bar{x}}_{\mu}d\bar{s}, \tag{2.5}$$

where the integral is over the world line of $\bar{e}$, $(x-\bar{x})^2$ is the square of the four-dimensional distance

$$(x-\bar{x})^2=\eta_{\mu\nu}(x^{\mu}-\bar{x}^{\mu})(x^{\nu}-\bar{x}^{\nu}), \tag{2.6}$$

and $\delta(\xi)$ is the Dirac delta function. The only contributions to the integral in Eq. (2.5) come from the two points where the null cone from x^{μ} intersects the world line of $\bar{e}$, i.e., from the retarded and advanced points on the world line.

Substituting from Eq. (2.5) into Eq. (2.3), we obtain

$$\delta\left[m\int ds+e\bar{e}\int\int\delta((x-\bar{x})^2)\dot{x}^{\mu}\dot{\bar{x}}_{\mu}dsd\bar{s}\right]=0. \tag{2.7}$$

Now the double integral in this expression is symmetric between the two particles e, m, $x^{\mu}(s)$ and $\bar{e}$, $\bar{m}$, $\bar{x}^{\mu}(\bar{s})$. This suggests making the whole action symmetric be-

[2] K. Schwarzschild, Nachr. Akad. Wiss. Göttingen Math. Physik. Kl. IIa **1903**, 128, 132, 245 (1903); H. Tetrode, Z. Physik **10**, 317 (1922); A. D. Fokker, *ibid.* **58**, 386 (1929); Physica **9**, 33 (1929); **12**, 145 (1932); J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **17**, 157 (1945); **21**, 425 (1949).

[3] J. W. Dettman and A. Schild, Phys. Rev. **95**, 1057 (1954); A. Schild, in *Proceedings of the International School of Physics "Enrico Fermi" Course 20* (Academic Press Inc., New York, 1963), pp. 69–115.

tween the particles, and considering

$$\delta\left[m\int ds+\bar{m}\int d\bar{s}+e\bar{e}\int\int\delta((x-\bar{x})^2)\dot{x}^\mu\dot{\bar{x}}_\mu ds d\bar{s}\right]=0. \quad (2.8)$$

Clearly, as long as we vary only the world line of e, the action principles (2.7) and (2.8) are equivalent, and both lead to the equations of motion (2.1), since the new term $\bar{m}\int d\bar{s}$ behaves like a constant under such variations. However, due to the symmetry of Eq. (2.8), we can also obtain the equations of motion of $\bar{e}$ in the time-symmetric electromagnetic field of e by varying the world line of $\bar{e}$.

Equation (2.8) is the Fokker action principle for a pair of point charges. The requirement that the action be stationary for variations of both world lines yields the equations of motion of each charge in the time-symmetric field of the other. The generalization to a system of any number of point charges is immediate, the action consisting of a sum of inertial terms of the form $m\int ds$, one for each particle, and of interaction terms of the form of the double integral in Eq. (2.8), one for each pair of particles.

The Lorentz invariance of the Fokker action implies that the variational equation (2.8) is satisfied identically for those variations which are induced by infinitesimal Lorentz transformations. This gives rise to ten identities and these identities, when combined with the equations of motion, take the form of ten conservation laws, four for the conservation of 4-momentum P^μ and six for the conservation of angular momentum $L^{\mu\nu}=-L^{\nu\mu}$. For a two-body system, they are as follows:

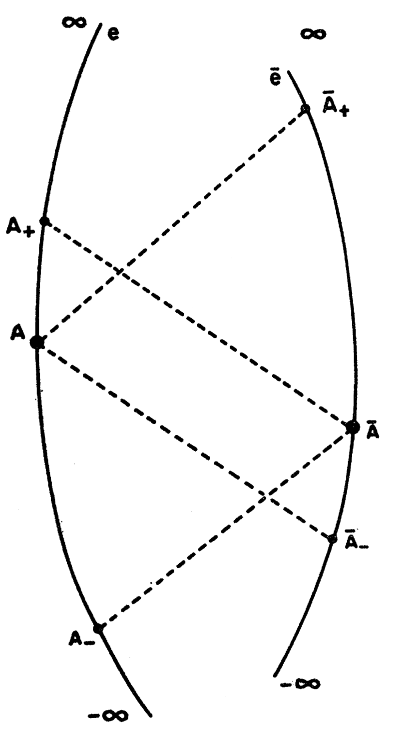

FIG. 2. Conservation laws.

Choose an event A on the world line of particle e and an event $\bar{A}$ on that of $\bar{e}$ (Fig. 2). Then the total linear momentum of the system of interacting particles is

$$P^\mu=[m\dot{x}^\mu+eA^\mu]_A+[\bar{m}\dot{\bar{x}}^\mu+\bar{e}\bar{A}^\mu]_{\bar{A}}+2e\bar{e}\left(\int_A^\infty\int_{-\infty}^{\bar{A}}-\int_{-\infty}^{A}\int_{\bar{A}}^{\infty}\right)\times\delta'((x-\bar{x})^2)(x^\mu-\bar{x}^\mu)\dot{x}^\nu\dot{\bar{x}}_\nu ds d\bar{s}. \quad (2.9)$$

The linear momentum is conserved in the sense that P^μ, as just defined, is independent of the choice of the two events A and $\bar{A}$ on the two world lines. The total angular momentum of the system about the origin is

$$\begin{aligned}L_{(0)}{}^{\mu\nu}=&[x^\mu(m\dot{x}^\nu+eA^\nu)-x^\nu(m\dot{x}^\mu+eA^\mu)]_A\\&+[\bar{x}^\mu(\bar{m}\dot{\bar{x}}^\nu+\bar{e}\bar{A}^\nu)-\bar{x}^\nu(\bar{m}^\mu\dot{\bar{x}}+\bar{e}\bar{A}^\mu)]_{\bar{A}}\\&+2e\bar{e}\left(\int_A^\infty\int_{-\infty}^{\bar{A}}-\int_{-\infty}^{A}\int_{\bar{A}}^{\infty}\right)\\&\times[\delta'((x-\bar{x})^2)(x^\nu\bar{x}^\mu-x^\mu\bar{x}^\nu)\dot{x}^\rho\dot{\bar{x}}_\rho\\&-\tfrac{1}{2}\delta((x-\bar{x})^2)(\dot{x}^\nu\dot{\bar{x}}^\mu-\dot{x}^\mu\dot{\bar{x}}^\nu)]ds d\bar{s},\end{aligned} \quad (2.10)$$

and again, this expression is independent of the choice of the events A and $\bar{A}$.

Because of the delta function and its derivative in the expressions for P^μ and $L_{(0)}{}^{\mu\nu}$, the total linear and angular momentum of the system is determined by finite portions of the world lines of the two particles. Thus, if A and $\bar{A}$ are chosen as shown in Fig. 2, the relevant portions are the arcs A_-AA_+ and $\bar{A}_-\bar{A}\bar{A}_+$, where the dashed lines are null lines.

A system with total linear momentum (P^μ is assumed to be time-like: $P_\mu P^\mu>0$) and total angular momentum always permits the definition of a relativistic center of mass, whose world line is a straight line parallel to the time-like vector P^μ. This is done as follows:

The total angular momentum $L_{(c)}{}^{\mu\nu}$ about a fixed event c^μ is defined by replacing x^μ and $\bar{x}^\mu$ by $x^\mu-c^\mu$ and $\bar{x}^\mu-c^\mu$ in the right-hand side of Eq. (2.10). We then find

$$L_{(c)}{}^{\mu\nu}=L_{(0)}{}^{\mu\nu}-c^\mu P^\nu+c^\nu P^\mu. \quad (2.11)$$

The world line of the center of mass of the system is now defined to consist of those events c^μ for which $L_{(c)}{}^{\mu\nu}$ has no components in the direction of P^μ, i.e., those events c^μ for which

$$L_{(c)}{}^{\mu\nu}P_\nu=0. \quad (2.12)$$

The solution is

$$c^\mu=(L_{(0)}{}^{\mu\nu}P_\nu/P^\alpha P_\alpha)+\lambda P^\mu, \quad (2.13)$$

where λ is arbitrary. These events c^μ clearly lie on a straight line parallel to P^μ, the world line of the center of mass.

In the center-of-mass frame, i.e., in the inertial frame where the world line of the center of mass of the system coincides with the time axis, so that $c^1=c^2=c^3=0$, we have

$$P^1=P^2=P^3=0\,, \tag{2.14}$$

$$L^{14}=L^{24}=L^{34}=0\,. \tag{2.15}$$

The only surviving components of the conserved quantities are the total energy

$$P^4=E\,, \tag{2.16}$$

and the space components of the total angular momentum about the origin

$$(L^{23},L^{31},L^{12})\,. \tag{2.17}$$

3. CIRCULAR MOTION OF POINT CHARGES

The results of a rigorous relativistic calculation will be given now for two point charges in circular motion interacting through their time-symmetric field.

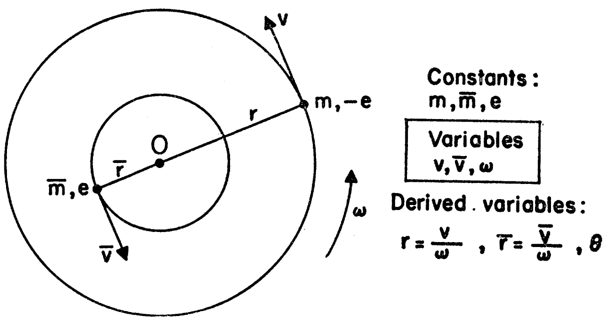

FIG. 3. Point charges in circular motion.

The two point charges are $-e$, m and e, $\bar{m}$, as shown in Fig. 3. We choose as independent variables describing the motion the speeds of the two charges v and $\bar{v}$ and the angular velocity ω of the system. The radii of the orbits are then derived variables given by $r=v/\omega$ and $\bar{r}=\bar{v}/\omega$. Another derived variable is the retardation angle θ, defined as the angle from O through which one particle turns during the time $\tau=\theta/\omega$ which it takes light from the other particle to reach it. The retardation angle θ is the positive root of the retardation relation which can be read off Fig. 4:

$$v^2+\bar{v}^2+2v\bar{v}\cos\theta-\theta^2=0\,. \tag{3.1}$$

The radial component of the equations of motion (2.1) of e is

$$\frac{m}{\omega}\frac{v}{(1-v^2)^{1/2}}=\frac{e^2}{(\theta+v\bar{v}\sin\theta)^3}[(v+\bar{v}\cos\theta)(1-v^2)(1-\bar{v}^2)+(v\theta+\bar{v}\sin\theta)(\theta+v\bar{v}\sin\theta)]\,, \tag{3.2}$$

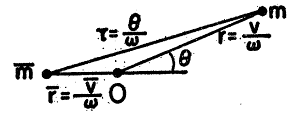

FIG. 4. The retardation relation.

and the radial equation of motion of the other charge is

$$\frac{\bar{m}}{\omega}\frac{\bar{v}}{(1-\bar{v}^2)^{1/2}}=\frac{e^2}{(\theta+v\bar{v}\sin\theta)^3}[(\bar{v}+v\cos\theta)(1-\bar{v}^2)(1-v^2)+(\bar{v}\theta+v\sin\theta)(\theta+v\bar{v}\sin\theta)]\,. \tag{3.3}$$

Because we have used half-retarded plus half-advanced electromagnetic fields, the components of the equations of motion tangential to the circular orbits are satisfied identically. Had we used purely retarded interactions, this would not be the case, and a stationary circular motion of two point charges would be impossible.

It is clear from the symmetry of the problem that the center O of the two circular orbits is the center of mass of the system. A calculation of P^μ, given by Eq. (2.9), confirms that $P^1=P^2=P^3=0$ and gives for the total energy of the system the simple expression

$$E=m(1-v^2)^{1/2}+\bar{m}(1-\bar{v}^2)^{1/2}\,. \tag{3.4}$$

The only nonzero component of angular momentum about O [Eq. (2.10)] is

$$L=L^{12}=e^2\frac{1+v\bar{v}\cos\theta}{\theta+v\bar{v}\sin\theta}\,. \tag{3.5}$$

Equation (3.4) shows that the binding energy of the two point charges is positive, i.e., $E<m+\bar{m}$. This contradicts a result of Sternglass,[4] who claims to obtain a relativistic state of an electron-positron pair in circular motion with energy close to the pion mass, even when the spins and magnetic moments of the particles are neglected.

The last argument depends on the choice (2.9) of the energy-momentum vector of the system, and this choice is not unique. However, it is unlikely that a different definition of energy and momentum can be given which is natural and which is finite for a system in periodic motion. Wheeler and Feynman[5] have shown how the canonical 4-momentum vector (2.9) can be obtained from a stress-energy tensor constructed from the electromagnetic fields associated with the particles. They also discuss another choice of stress-energy tensor, due to Frenkel, which leads to a 4-momentum differing from the canonical 4-momentum by the term

$$P_{Fr}{}^{\mu}-P^\mu=e\bar{e}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\epsilon(x,\bar{x})\delta'((x-\bar{x})^2)(x^\mu-\bar{x}^\mu)\dot{x}^\nu{}_\nu\dot{\bar{x}}ds d\bar{s}\,, \tag{3.6}$$

[4] E. J. Sternglass, Phys. Rev. **123**, 391 (1961).
[5] J. A. Wheeler and R. P. Feynman, Rev. Mod. Phys. **21**, 431 (1949).

where

$$\epsilon(x,\bar{x}) = +1 \quad \text{for} \quad x^4 < \bar{x}^4$$
$$= -1 \quad \text{for} \quad \bar{x}^4 < x^4. \tag{3.7}$$

Because of the infinite integrations in Eq. (3.6), it is clear that the additional term is either zero or infinite for a periodic motion of the system. Thus, for periodic motions the Frenkel 4-momentum either coincides with the canonical 4-momentum or gives infinite results. In our case of point charges in circular motion, the right-hand side of Eq. (3.6) vanishes, so that the Frenkel 4-momentum also leads to the energy given by Eq. (3.4).

Our system, characterized by Eqs. (3.1) to (3.5), can now be quantized by putting $L = n\hbar$. For either positronium (e electronic charge, $m = \bar{m}$ electron mass) or hydrogen (e, m electronic charge and mass, $\bar{m}$ proton mass), the resulting quantized motions are all nonrelativistic. They are the usual Bohr motions with small corrections for retardation and other relativistic effects and, in the case of hydrogen, with small corrections for the motion of the nucleus.

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Coherent and Incoherent States of the Radiation Field*

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Methods are developed for discussing the photon statistics of arbitrary radiation fields in fully quantum-mechanical terms. In order to keep the classical limit of quantum electrodynamics plainly in view, extensive use is made of the coherent states of the field. These states, which reduce the field correlation functions to factorized forms, are shown to offer a convenient basis for the description of fields of all types. Although they are not orthogonal to one another, the coherent states form a complete set. It is shown that any quantum state of the field may be expanded in terms of them in a unique way. Expansions are also developed for arbitrary operators in terms of products of the coherent state vectors. These expansions are discussed as a general method of representing the density operator for the field. A particular form is exhibited for the density operator which makes it possible to carry out many quantum-mechanical calculations by methods resembling those of classical theory. This representation permits clear insights into the essential distinction between the quantum and classical descriptions of the field. It leads, in addition, to a simple formulation of a superposition law for photon fields. Detailed discussions are given of the incoherent fields which are generated by superposing the outputs of many stationary sources. These fields are all shown to have intimately related properties, some of which have been known for the particular case of blackbody radiation.

I. INTRODUCTION

FEW problems of physics have received more attention in the past than those posed by the dual wave-particle properties of light. The story of the solution of these problems is a familiar one. It has culminated in the development of a remarkably versatile quantum theory of the electromagnetic field. Yet, for reasons which are partly mathematical and partly, perhaps, the accident of history, very little of the insight of quantum electrodynamics has been brought to bear on the problems of optics. The statistical properties of photon beams, for example, have been discussed to date almost exclusively in classical or semiclassical terms. Such discussions may indeed be informative, but they inevitably leave open serious questions of self-consistency, and risk overlooking quantum phenomena which have no classical analogs. The wave-particle duality, which should be central to any correct treatment of photon statistics, does not survive the transition to the classical limit. The need for a more consistent theory has led us to begin the development of a fully quantum-mechanical approach to the problems of photon statistics. We have quoted several of the results of this work in a recent note,[1] and shall devote much of the present paper to explaining the background of the material reported there.

Most of the mathematical development of quantum electrodynamics to date has been carried out through the use of a particular set of quantum states for the field. These are the stationary states of the non-interacting field, which corresponds to the presence of a precisely defined number of photons. The need to use these states has seemed almost axiomatic inasmuch as nearly all quantum electrodynamical calculations have been carried out by means of perturbation theory. It is characteristic of electrodynamical perturbation theory that in each successive order of approximation it describes processes which either increase or decrease the number of photons present by one. Calculations performed by such methods have only rarely been able to deal with more than a few photons at a time. The

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[1] R. J. Glauber, Phys. Rev. Letters 10, 84 (1963).